

# Quantum chaos inside Black Holes

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We show how semiclassical black holes can be reinterpreted as an effective geometry, composed of a large ensemble of horizonless naked singularities (eventually smoothed at the Planck scale). We call these new items *frizzyballs*, which can be rigorously defined by euclidean path integral approach. This leads to interesting implications about information paradoxes. We demonstrate that infalling information will chaotically propagate inside this system before going to the full quantum gravity regime (Planck scale).

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## 1. INTRODUCTION

Black holes' information paradox is an insidious field of discussions. Bekenstein-Hawking semiclassical approach [1, 2] suggests that, from a pure state falling into a Black Hole, a highly mixed final state is obtained. This apparently implies that information is lost and unitarity is violated. On the other hand AdS/CFT correspondence seems to suggest that information is preserved in BH [3–5]<sup>1</sup>. Complementarity in Black Hole physics [6, 7] is not enough to solve these problems: it leads to the firewall paradox [8, 9]<sup>2</sup>. So, surely there is something more that we are missing in the complementarity picture. Information problem is "a battle field" of many different ideas and interpretations of Black holes [12–39]<sup>3</sup>.

On the other hand, we know that in laboratory, entangled pairs of two photons can be prepared from ingoing pure photons. This is the so called *spontaneous parametric down-conversion* (SPDC) (see [48] for a review in these subjects): it consists of a nonlinear crystal pumped with intense coherent light. What happens is a three-wave mixing mechanism using (the lowest order) nonlinear susceptibility of non-linear birefringent crystals. As well understood, this is not a violation of quantum mechanic principles: the evolution from a pure state to a mixed one is just due to a quantum decoherence effect, *i.e* the infalling information is entangled with the rest of the crystal through interactions with it. In this case a wave function or an S-matrix approach describing the dynamics of the infalling information is not useful: a density matrix approach is preferred. The density matrix associated to the infalling pure state has a non-unitary evolution described by a Liouville equation. Following such an analogy, we would like to suggest that a mechanism similar to SPDC converts infalling pure states into out-falling mixed ones inside quantum black hole, without violating unitarity at fundamental level. In this case, a quantum decoherence mechanism is realized. On the other hand, such a mechanism seems not possible in a semiclassical black hole: its geometry is smoothed and a particle infalling into a semiclassical BH will not experience drastic changes because of the equivalence principle.

In this paper, we suggest that a semiclassical black hole geometry is an approximated effective solution: it is a system of a large number of horizonless geometries, in semiclassical regime. Such a system can be imagined as a sort of "frizzy-ball": space-time asperities will differentiate the ideal semiclassical black hole by the complicated system of metrics. A "frizzy black hole" has an approximated classical event horizon, approximately emitting Bekenstein-Hawking radiation. How much "frizzy" with respect to a Semiclassical Black hole can be rigorously defined by the departure of its emitted radiation with respect to a Bekenstein-Hawking thermal one, as we will discuss later. However, also a frizzyball emits a mixed state rather than a pure one for  $t \ll t_{BH}$ .<sup>4</sup>

This suggestion has important implications in the evolution of infalling information. Let us consider an initial pure state infalling into a frizzy black hole. What one could expect is that initially this just "feels" an approximately

<sup>1</sup> An intriguing issue regarding holography is the locality. Recent discussion on effective non-local field theories were discussed in [53, 58]

<sup>2</sup> Let us note that for extended theories of gravity, a problem could also be the presence of instabilities just in the classical formulation of the theory. See [50] for a discussion of external geodetic instabilities around BH or stars, in a class of massive gravity models.

<sup>3</sup> The approach considered in [12–16] can be related to another class of classicalons studied in [49, 51, 52, 54–57, 59–61].

<sup>4</sup> However, the effect of environmental radiation infalling into BH, such as CMB, was studied in [36]. They seem to demonstrate that environmental radiation can relevantly affect and suppress the emission of Bekenstein-Hawking radiation. Also in frizzy-balls, this effect is expected to be similar.

smoothed semiclassical space-time. In fact the quantum wave function of an infalling particle has a De Broglie wavelength that is much larger than asperities' scales. However, inevitably, the infalling wave will start to be blueshifted, so that it will start to "resolve" more and more the asperities of the non-trivial topology. At this point, the infalling wave will start to be scattered back and forth by the asperities, before going to the Planck scale. At this stage, information will start to be chaoticized inside the system<sup>5</sup>. This system can be thought as a wave function scattering on a *quantum Sinai billiard*!<sup>6</sup>

What one will expect is that the initial probability will be fractioned into two contributions. In fact, a part of the initial probability density will "escape", emitted as quasi-thermal radiation, by the system while a part will remain "trapped" forever in the system because of back and forth scatterings. This can be easily understood by a classical chaotic mechanics point of view. In fact, the definition of a classical chaotic scatterings of a particle is the following: a classical mechanics' scattering problem in which the incident particle can be trapped ideally forever in a class of classical orbits; but the periodic orbits are unstable saddle solutions and their number grows exponentially with time. Chaotic scatterings have a high sensitivity to the initial conditions manifesting itself in a fractal chaotic invariant set, which is also called *chaotic saddle* [44]. Energy shells closed to the chaotic saddle energy shell will continue to be chaotic.<sup>7</sup> In our case, periodic orbits will be forever trapped in back and forth scatterings among the the space-temporal Sinai biliard. As generically happening in classical chaotic scatterings' problems, these trajectories will necessary exist in the phase space of the system<sup>8</sup>. From Classical chaotic scatterings' one can get the main feature of the quantum semiclassical chaotic problem associated and about semiclassical periodic orbits. So, because of multiple diffractions and back and forth scatterings, one will also expect that the resultant wave function is "chaotized" by the system: the total wave function is a superposition of the initial one plus all the spherical ones coming from each "scatterators". A part of the initial infalling information will be trapped "forever" in the system, *i.e* for all the system life-time. In order to describe the evolution of the infalling informations, a quantum mechanical approach based on wave functions is not useful, in this system. A wave functions' approach can be substituted by a quantum statistical mechanics' approach in terms of density matrices. From the point of view of a Quantum field theory, a S-matrix approach is not useful in this case, even if "fundamentally true": in order to calculate  $\langle in|S|out \rangle$  (*in* is the in-going plane wave, where *out* is the out-going result), one has to get unknown informations on the precise geometric configuration inside the system and about the trapped information state inside it. Such a system can emit a quasi thermalized mixed information state without losing any informations at fundamental level. In other words, we suggest that the space-time non-trivial topology prepares an entangled state as well as an experimental apparatus can prepare an entangled state by an initial pure state. In our case, the effect will also be dramatically efficient: thank to quantum field theory' interactions in the lagrangian density functional, n-wave mixings will occur inside our system. Thinking about the ingoing state as a collection of coherent quantum fields, these will be scattered into the system and, they will meet each others inside "the trap", they will scatter each others, coupled by lagrangian interactions. A complicated cascade of hadronic and electromagnetic processes is expected. For example, these will produce a large amount of neutral pions, that will electromagnetically decay into two entangled photons  $\pi^0 \rightarrow \gamma\gamma$  ( $\tau \simeq 10^{-16}$  s in the rest frame). However, also from only one plane wave infalling in the system, the final state emitted by the system will be a mixed state: this is just an effect of the information losing inside the system because of trapped chaotic zones inside. This phenomena is a new form of quantum decoherence induced by the space-time topology. Usually, quantum decoherence is the effective losing of infalling informations in a complex system, like coherent light pumped in a non-linear crystal. In this case, the complex topology of space-time catalyzes the effective losing of information.

In our quantum chaotic system, we will not have any information paradoxes or firewalls. In fact, infalling pure information is converted to a mixed thermal state during  $t \ll t_{BH}$  because of a quantum decoherence inside the space-

<sup>5</sup> For different applications of chaos theory in information paradox see [37, 40–42].

<sup>6</sup> Quantum chaos studies the relationship between classical chaos and quantum mechanics. Quantum Sinai Billiard is a well known example of quantum chaotic system. For a general overview in quantum chaos see [43].

<sup>7</sup> An example in simpler systems are Kolmogorov-Arnold-Moser (KAM) elliptic islands, that contain stable periodic orbits. KAM stable periodic orbits undergo to chaotic bifurcations, rupturing the smoothed topology of the invariant set [45]. As usually happening for chaotic saddles, KAM islands are surrounded by a layer of chaotic trajectories. Another typical example is the hyperbolic set of hyperbolic unstable trajectories: solutions are exponentially growing or decreasing but the number of directions are constants of motion.

<sup>8</sup> Our problem is nothing but a complication with respect to a simpler and well known example of classical chaotic scattering problem: a 2d classical elastic scattering of a particle on a system of N fixed disks of radius  $a$  [46]. In this simple problem, kinetic energy is assumed to be conserved, *i.e* no any dissipations are considered. For one disk the problem is trivially un-chaotic: the differential cross section is just  $\frac{d\sigma}{d\theta} = \frac{a}{2} |\sin \frac{\theta}{2}|$  for  $\theta$  in the range  $[-\pi, \pi]$ ; and no trapped periodic orbit are possible. However, with two disks, an unstable periodic orbit is the one bouncing back and forth forever among the two disks. With the increasing of the number of disks one can easily get that the number of trapped periodic orbit will exponentially increase. For example, as shown in [47], in a three disks' system, the number of unstable periodic orbits proliferate as  $2^n$  where  $n$  is the number of bounces in unit of the period. if the radius is the distance among the next neighboring disk is  $R > 2.04822142 a$ .

time Sinai biliard. The evolution of this state is apparently non-unitary, but unitarity is fundamentally preserved: the lost information is trapped in chaotic zones inside the biliard. However, it seems that such a system cannot really hide information "forever": it has approximately the same Hawking's radiation (same thermal entropy) of BH for  $t \ll t_{BH}$ , so that it will completely evaporate after a certain time. As a consequence, during the final evaporation, the hidden information cannot be trapped anymore and it is re-given to the external environment in a "final information burst". So, CPT is "apparently" violated for a time  $t < t_{BH}$  in the external environment, but, after the final evaporation, CPT again manifests its conservation<sup>9</sup>. Such a phenomena is a sort of space-time phase transition, defined as a transition of the space-time topology itself: the frizzly space-time will transmute to a Minkowski like one. The trapped probability density is expected to be linearly dependent by the number of asperities as  $\rho \sim N_s e^{-\Gamma T}$ , where  $\Gamma$  is proportional to effective average deepness of systems of asperities trapping  $\rho$ . (This can be easily estimated in WKB approach). On the other hand, the number of asperities are sustained by the Black hole mass, in turn decreasing with the temperature as  $dM/dT = -1/8\pi T^2$ . As a consequence,  $\rho$  is approximately described by a simple differential equation  $d\rho(T)/dT \sim -\frac{1}{T^2} e^{-\Gamma T}$ <sup>10</sup>

The paper is organized as follows: In Section 2, we argument how a system of N horizonless singularities can recover a Semiclassical BH state; in Section 3 we will discuss the problem the problem of chaotic scatterings of matter infalling in a frizzlyball, in Section 4 we show our conclusions.

## 2. THE PATH-INTEGRAL APPROACH

In this section, we will give a path-integral formulation of our problem. We will precisely define what is a horizonless "frizzly-ball" with respect to a semiclassical euclidean black hole.

In general, the path integral over all euclidean metrics and matter fields is

$$Z_E = \int \mathcal{D}g \mathcal{D}\phi e^{-I[g,\phi]} \quad (1)$$

where  $g$  is the euclidean metric tensor. In semiclassical approach the relevant generally covariant lagrangian [63] is

$$I_E = - \int_{\mathcal{M}} \sqrt{g} d^4x \left( \mathcal{L}_m + \frac{1}{16\pi} R \right) + \frac{1}{8\pi} \int_{\partial\mathcal{M}} \sqrt{h} d^3x (K - K^0) \quad (2)$$

(we use  $G_N = 1$ ), where  $K$  is the trace of the curvature induced on the boundary  $\partial\mathcal{M}$  of the region  $\mathcal{M}$  considered,  $h$  is the metric induced on the boundary  $\partial\mathcal{M}$ ,  $K^0$  is the trace of the curvature induced imbedded in flat space. The last term is a contribution from the boundary. As usually done in semiclassical WKB approach, one can perturb matter fields and metric as  $\phi = \phi_0 + \tilde{\phi}$  and  $g = g_0 + \tilde{g}$ , so that

$$I[\phi, g] = I[\phi_0, g_0] + I_2[\tilde{\phi}, \tilde{g}] + \text{higher orders}$$

$$I_2[\tilde{\phi}, \tilde{g}] = I_2[\tilde{\phi}] + I_2[\tilde{g}]$$

$$\log Z = -I[\phi_0, g_0] + \log \int \mathcal{D}\tilde{\phi} \mathcal{D}\tilde{g} e^{-I_2[\tilde{g}, \tilde{\phi}]} \quad (3)$$

In an Euclidean Schwarzschild solution, the metric has a periodicity in time  $i\beta$ , where

$$\beta = T^{-1} = 8\pi M$$

with  $T, M$  BH temperature and mass. This metric has a form

$$ds_E^2 = \left(1 - \frac{2M}{r}\right) d^2\tau + \left(1 - \frac{2GM}{r}\right) dr^2 + r^2 d\Omega^2 \quad (4)$$

<sup>9</sup> Another apparent violation of CPT induced in neutron-antineutron system by a new interaction was discussed in [62].

<sup>10</sup> In order to avoid any confusions, let us clarify what we mean for (quasi) "forever" trapped information. In fact, one observer outside and one inside the frizzlyball will disagree about BH lifetime. The disagreement depends by the BH mass, but typically for  $M > M_\odot$  the observer inside the frizzlyballs will measure  $t_{BH} \ll 1$  yr. However, this short life-time is enough for an efficient chaotization: particles' waves get high kinetic energy by the gravitational field, *i.e* we are considering a relativistic Sinai Biliard, with very fast quantum interactions among fields.

but this can be conveniently rewritten in terms of a new variable

$$x = 4M\sqrt{1 - \frac{2M}{r}}$$

$$ds_E^2 = \left(\frac{x}{4M}\right)^2 + \left(\frac{r^2}{4M^2}\right)^2 dx^2 + r^2 d\Omega^2 \quad (5)$$

that it is free by the mathematical singularity in  $r = 2M$ , while the Euclidean time  $\theta$  is angular variable with period  $\beta = 8\pi M$ . As a consequence, the boundary  $\partial M$  has a topology  $S^1 \times S^2$  at a certain fixed radius  $r_0$ . In a stationary-phase point approximation, the path integral becomes just a partition function of a canonical ensemble, with temperature  $T = \beta^{-1}$ . In semiclassical approximation, the dominant contribution to the path integral in a Euclidean background is

$$Z_{ES} = e^{-\frac{\beta^2}{16\pi}} \quad (6)$$

Such a term is coming by surface integrals in the action.

(6) is thermo-dynamically related to the average energy as

$$\langle E \rangle = -\frac{d}{d\beta}(\log Z) = \frac{\beta}{8\pi} \quad (7)$$

while  $\log Z$  is usually defined in statistical mechanic as

$$\log Z = -\frac{F}{T} \quad (8)$$

But the entropy is related to the average and free energy as

$$S = \beta(F - \langle E \rangle) \quad (9)$$

so that one can obtain the Hawking's entropy

$$S = \beta(\log Z - \frac{d}{d\beta}(\log Z)) = \frac{\beta^2}{16\pi} = \frac{1}{4}A \quad (10)$$

Let us remember that the physical interpretation of the semiclassical BH path integral is that a BH is confined in a box with a fixed size, and it is consider in thermal equilibrium with its own Hawking radiation, at a constant temperature  $T$ .

After this short review, let us give the definition of frizzyball.

**Def:** let us consider a generic system of  $N$  horizonless singularities (suppose to be eliminated at the Planck scale) inside a box with a surface  $\partial M$ . This system is a *frizzyball* if it satisfies the following hypothesis:

i) The  $N$  horizonless singularities are in thermal equilibrium with the box, and a formal definition of partition functions  $Z_I$  for each metric tensor  $g^{I=1,\dots,N}$  exists.

ii) In semiclassical approximation, the leading order of the total partition function associated to this system is the product of the single partition function:

$$Z_{TOT} = \prod_{I=1}^N Z_I \quad (11)$$

This corresponds to consider the total entropy in the system as the sum of entropies associated to each naked singular geometries, *i.e*

$$\log Z_{TOT} = \sum_{I=1}^N \log Z_I \quad (12)$$

The physical interpretation is that the intergeometries' interactions are negligible with respect to the temperature of the system inside the box <sup>11</sup>.

iii) The total average partition function is

$$\langle Z_{TOT} \rangle = e^{-\frac{\beta^2}{16\pi} - \frac{\sigma_\beta^2}{16\pi}} = Z_{ES} e^{-\frac{\sigma_\beta^2}{16\pi}} \quad (13)$$

where  $\sigma_\beta$  is the variance of  $\beta$ -variable in the system, and it is assumed to be very small even different from zero. In fact this parametrizes the small deviations of the semiclassical frizzy-ball with respect to semiclassical BH, . *i.e* the local not perfect smoothness of the frizzy geometry. Eq.(13) is understood considering deviations  $\beta + \delta\beta$ , with  $\delta\beta \ll \beta$ :

$$e^{-\frac{\beta^2 + 2\beta\delta\beta + O(\delta\beta^2)}{16\pi}} = Z_{ES} \times e^{\frac{\beta\delta\beta}{8\pi} + O(\delta\beta^2)}$$

and assuming  $\langle \delta\beta \rangle = 0$ .

Eq.13 leads to the entropy

$$\langle S \rangle = \frac{\beta^2}{16\pi} + \frac{\sigma_\beta^2}{16\pi} \quad (14)$$

Let us note that even if a small correction to the Bekenstein-Hawking entropy is predicted, the out-going radiation is expected to be a mixed state also for this system.

The next non-trivial step is to demonstrate the mathematical consistence of the definition of a frizzy-ball, *i.e* if the three hypothesis not lead to any contradictions. In semiclassical approximation, the existence of a frizzy-ball is related to the following identity:

$$-I[g_0, \phi_0] + \log \int \mathcal{D}\tilde{\phi} e^{-I_2[g_0, \phi]} + \log \int \mathcal{D}\tilde{g} e^{-I[\tilde{g}]} = -\sum_J I[g_0^J, \phi_0] + \sum_J \left[ \log \int \mathcal{D}\tilde{\phi} e^{-I_2[g_0^J, \phi]} + \log \int \mathcal{D}\tilde{g}^J e^{-I[\tilde{g}^J]} \right] \quad (15)$$

where  $g_0$  is the Euclidean Schwarzschild metric tensor while  $g_0^J$  are the Euclidean metric tensor of  $J = 1, \dots, N$  geometries. This leads to the following classical relations

$$g_0 = \left( \sum_J \sqrt{g_0^J} \right)^2 + \text{higher orders} \quad (16)$$

$$\sqrt{g_0} R(g_0) = \sum_J \sqrt{g_0^J} R(g_0^J) + \text{higher orders} \quad (17)$$

while quantum fluctuations are

$$I_2[\tilde{g}] = \sum_J I_2[\tilde{g}^J] + \text{higher orders} \quad (18)$$

### 3. "BLACK" CHAOTIC SINAI BILIARD

In this section, we will argue the apparently information lost in a system of horizonless singularities. In subsection 3.1 and 3.2 we will use a non-relativistic approach. This approximation is not fully justified in our realistic problem, as well as a non-relativistic quantum mechanic approach to scattering problems in particle physics. However, one could retain useful to discuss simplified problems rather than the realistic one, in order to get easier relevant chaotic aspects. In subsection 3.3 we will formally comment our problem from a QFT point of view.

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<sup>11</sup> This approximation seems not compatible with chaotization of information inside the frizzyball. Infact, chaotization is related to an exchange of matter among the geometries. However, this apparent contradiction is avoid in frizzyball system. In fact, if the net exchange of heat among the geometries is negligible with respect to the thermal energy in the box, as expected in a system in thermal equilibrium, this approximation will be rightly applied. It corresponds to  $S_{int} \simeq 0$ . As regards gravitational interactions among the metrics, this is strongly suppressed in semiclassical regime.

### 3.1. Classical chaotic scattering on a Space-time Sinai Biliard

The classical chaotic scattering of a particle on an box of horizonless singularities is characterized by a classical Hamiltonian system  $\dot{\mathbf{r}} = \partial H / \partial \mathbf{p}$  and  $\dot{\mathbf{p}} = -\partial H / \partial \mathbf{r}$  with an initial condition  $x_0 = (\mathbf{r}_0, \mathbf{p}_0)$  in the space of phase. In particular, one can define  $N$  Hamiltonian systems for each geometry, describing the motion of the particle on each of  $N$  geometries. Clearly, one can obtain similar systems by the geodesic equations  $\ddot{x}^\mu + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta = 0$  of the particles in each  $g_{\mu\nu}^I$  metrics, *i.e* the propagation of the particle on the  $I$ th hypersurfaces. The effective Hamiltonian obtained for the propagation on a  $I$ th metric is

$$H_I = \frac{1}{2m} p_i g^{ij} p_j = \frac{p^2}{2m} + V$$

in non-relativistic regime, where  $V$  is an effective potential depending on the  $I$ th geometry. The solution of such a system will be determined by a trajectory  $x_t = \phi^t(x_0)$  solving the Chauchy problem of classical mechanics. In this case, we will expect a proliferation of trapped periodic unstable trajectories, as anticipated in the introduction, because of an infinite back and forth scattering among the  $N$  geometries.

Let us define the action of the classical problem:

$$S(E) = - \int_{\Sigma} \mathbf{r} \cdot \mathbf{p} \quad (19)$$

where  $\Sigma$  is the energy shell  $H = E$  where a scattering orbit is sited. The time delay is defined as

$$\mathcal{T}(E) = \frac{\partial S}{\partial E} \quad (20)$$

If the impact parameters of the initial orbits  $\rho$  has a probability density  $w(\rho)$ , the probability density conditioned by energy  $E$  of the corresponding time delays is

$$P(\tau|E) = \int d\rho w(\rho) \delta(\tau + \mathcal{T}(\rho|E)) \quad (21)$$

where the condition "corresponding time delays" is encoded in the integral though the Dirac's delta. (21) is useful to describe the escape of the particle from the trapped orbits' zone. Inspired by  $N$  disks' problems studied in literature, an hyperbolic invariant set is expected to occur. In this case the decays' distribution rate is expected to exponentially decrease, *i.e*

$$\lim_{t \rightarrow \infty} \frac{P(\tau|E)}{t} = -\gamma(E) \quad (22)$$

On the other hand, for non-hyperbolic sets, like KAM elliptic islands, power low decays are generically expected  $P(t|E) \sim 1/t^\alpha$ , where  $\alpha$  depends by the articular density of trapped orbits.

Now, let us discuss the time delays in our system of horizonless geometries. If unstable periodic orbit exists in our scattering problem, eq.(20) will have  $\rho$ -poles, *i.e* it becomes infinite for precise initial impact parameters  $\rho$ . Let starts with the simplest case of a scattering on one geometry. Let us suppose that this geometry has  $n$ -periodic directions. For example, a conic singularity has a periodic direction around its axis and so on. In this case, the integral (20) has a couple of asymptotic divergent direction along each paths  $x_t^{\theta^{(n)}} = (\theta^{(n)}, p_\theta^{(n)})$ , where  $\theta^{(n)}$  are the periodic variables and  $p_\theta^{(n)}$  are their conjugated momenta. The particle will be infinitely trapped in these paths if and only if its initial incident direction is parallel to one of the periodic directions  $\theta^{(n)}$ .

Now let us complicate the problem considering two geometries. In these case the number of divergent asymptotes of  $\mathcal{T}$  correspond to three couples: i) cycles around the first geometry, ii) cycles around the second geometry, iii) trapped back and forth trajectories between the two geometries. As a consequence, just in this case the number of trapped trajectories is enormously growing.

One can easily get that for a  $N$  number of horizonless singularities the number of the divergent asymptotes for the time-delay function will proliferate. These divergent asymptotes are connected to the fractal character of the invariant set. A geometric way to see the problem is the following: one can consider a  $2\nu - 2$  Poincaré surface with section in the Hamiltonian flown on a fixed energy surface, where  $\nu$  is the number of degree of freedom of the system. In our case, we consider a 4d Poincaré surface. The time-delay of the orbit necessary to go-out from the cones at large enough distances is  $\mathcal{T}_\pm(\rho|E)$ , for every initial Chauchy condition in the Poincaré section.  $\mathcal{T}_+(\rho|E) \rightarrow \infty$  stable surfaces of orbits trapped forever. On the other hand,  $\mathcal{T}_- \rightarrow \infty$  on unstable manifolds of orbits

In other words,  $|\mathcal{T}_-(\rho|E)| + |\mathcal{T}_+(\rho|E)|$  is a localizer functions for the fractal set trapped trajectories. Let us remind the definition of sensitivity to initial conditions, defined by the Lyapunov exponents

$$\lambda(x_0|\delta x_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \frac{|\delta x_t|}{|\delta x_0|} \quad (23)$$

where  $\delta x_{0,t}$  are infinitesimal perturbation of the initial condition  $x_0$  and the resultant orbit  $x_t$ . In general, the Lyapunov exponents depend on the initial perturbation and on the orbit perturbation. However,  $\lambda$  becomes unsensible by the orbit in ergodic invariant sets

These sets are characterized by the following hierarchy of Lyapunov exponents in a system with  $\nu$ -degrees of freedom:

$$0 = \lambda_\nu \leq \lambda_{\nu-1} \leq \dots \leq \lambda_2 \leq \lambda_1 \quad (24)$$

while

$$0 = \lambda_{\nu+1} \geq \dots \geq \lambda_{2\nu} \quad (25)$$

In a Hamiltonian system, the symplectic flows of the Hamiltonian operator implies that

$$\sum_{k=0}^{2\nu} \lambda_k = 0$$

and

$$\lambda_{2\nu-k+1} = -\lambda_k$$

where  $k = 1, 2, \dots, 2\nu$ . In our case, the number of degree of freedom is  $\nu = 3$ , so that the number of independent Lyapunov's exponents characterizing the chaotic scattering is three.

The exponentially growing number of unstable periodic trajectories inside the invariant set is characterized by a topological number

$$h = \lim_{t \rightarrow \infty} \frac{1}{t} \ln(\mathcal{N}\{\tau_o \geq t\}) \quad (26)$$

where  $\mathcal{N}$  is the number of periodic orbits of period minor than  $t$ ,  $\tau_o$  is the periodic orbit time. Such a number is the so called topological entropy

$h > 0$  if the system is chaotic while  $h = 0$  if non-chaotic. For a system like a large box of horizonless geometries, this number will be infinite. Such a number will diverge just with only three geometries as happen just in a system of three 2d-disks.

In our system, as for disks, a hyperbolic invariant set or something of similar is expected. For this set  $\delta V$  small volumes are exponentially stretched by

$$g_\omega = \exp\left\{\sum_{\lambda_k > 0} \lambda_k t_\omega\right\} > 1 \quad (27)$$

because of its unstable orbits; where  $t_\omega$  is the time interval associated to the periodic orbit of period  $n$ , *i.e* to the symbolic dynamic  $\omega = \omega_1 \dots \omega_n$  corresponding to all the nonperiodic and periodic orbits remaining closed in a  $\delta V$  for a time  $t_\omega$ . Using (27), one can weight the probabilities for trapped orbits as

$$\mu_\alpha(\omega) = \frac{|g_\omega|^{-\alpha}}{\sum_\omega |g_\omega|^{-\alpha}} \quad (28)$$

This definition is intuitively understood: a highly unstable trajectory with  $g_\omega \gg 1$  is weighted as  $\mu_\alpha \simeq 0$ . The definition (28) is normalized  $\sum_\omega \mu_\alpha(\omega) = 1$ . With  $\alpha = 1$  we recover the ergodic definition for the Hamiltonian system.

An intriguing question will be if one can determine the Hausdorff dimension of the fractal sets for our box of cones. In principle, the answer is yes, but in practice the problem seems really hard to solve. In order to get the problem let us define the Ruelle topological pressure

$$P(\alpha) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \sum_{\omega, t < t_\omega < t + \Delta t} |g_\omega|^{-\alpha} \quad (29)$$



Ruelle topological pressure is practically independent by  $t, \Delta t$  for a large  $\Delta t$ . The Ruelle topological pressure has a series of useful relations:

- 1)  $P(\alpha_1 + \alpha_2) \leq P(\alpha_1) + P(\alpha_2)$
- 2)  $P(0) = h$ , *i.e* for  $\alpha = 0$  the Ruelle topological pressure is just equal to the topological entropy.
- 3)  $P(1) = -\gamma$ , *i.e* for  $\alpha = 1$  the Ruelle topological pressure is just equal to the escape rate.
- 4) The Ruelle topological pressure is connected to Lyapunov's exponents as

$$\frac{dP}{d\beta}(1) = -\lim_{t \rightarrow \infty} \sum_{\omega, t < t_\omega < t + \Delta t} \mu_1(\omega) \ln |g_\omega| = - \sum_{\lambda_k > 0} \lambda_k$$

The last relation is the one connecting the Ruelle topological pressure with the Hausdorff dimension  $d_H$ : 5)  $P(d_H) = 0$ . The Hausdorff dimension of a system with  $\nu$  d.o.f is bounded as  $0 \leq d_H \leq \nu - 1$  for the subspace of unstable directions, while a corresponding set of stable directions has exactly the same dimension of the previous one. Let us note that for a system with  $\nu = 1$  the Hausdorff dimension will collapse to  $d_H = 0$ , *i.e* no chaotic dynamics. In our case,  $0 \leq d_H \leq 2$  and in principle it can be founded as a root of the Ruelle topological pressure.

### 3.2. Semiclassical chaotic scattering on a Space-time Sinai Biliard

A natural approach to quantum chaotic scattering can be to consider a semiclassical approach correspondent to the classical chaotic problem. In Semiclassical approach, the main aspects of fully classical limit are remaining: trapped periodic orbits, invariant sets and so on. In semiclassical approach one can generalize the classical notion of time delay for a semiclassical quantum system.

Let us remind, just to fix our conventions, that  $\psi_t(\mathbf{r})$  is obtained by an initial  $\psi_0(\mathbf{r}_0)$  by the unitary evolution

$$\psi_t(\mathbf{r}) = \int d\mathbf{r}_0 K(\mathbf{r}, \mathbf{r}_0, t) \psi_0(\mathbf{r}_0) \quad (30)$$

where  $K$  is the propagator, represented as a non-relativistic Feynman path integral as

$$K(\mathbf{r}, \mathbf{r}_0, t) = \int \mathcal{D}\mathbf{r} e^{\frac{i}{\hbar} I} \quad (31)$$

where

$$I = \int_0^t dt L(\mathbf{r}, \dot{\mathbf{r}})$$

I the action and  $L$  the lagrangian of the particle. The semiclassical limit is obtained in the limit

$$I = \int_0^t [\mathbf{p} \cdot d\mathbf{r} - H d\tau] \gg \hbar$$

so that the leading contribution to the path integral is just given by classical orbits. The corresponding WKB propagator has a form

$$K_{WKB}(\mathbf{r}, \mathbf{r}_0, t) \simeq \sum_n \mathcal{A}_n(\mathbf{r}, \mathbf{r}_0, t) e^{\frac{i}{\hbar} I_n} \quad (32)$$

where we are summing on all over the classical orbits of the system, while amplitudes  $\mathcal{A}_n$  are

$$\mathcal{A}_n(\mathbf{r}, \mathbf{r}_0, t) = \frac{1}{(2\pi i \hbar)^{\nu/2}} \sqrt{|\det[\partial \mathbf{r}_0 \partial \mathbf{r}_0 I_n[\mathbf{r}, \mathbf{r}_0, t]]|} e^{-\frac{i\pi h_n}{2}} \quad (33)$$

where  $h_n$  counts the number of conjugate points along the n-th orbit.

From (53), the probability amplitude is related to Lyapunov exponents as

$$|\mathcal{A}_n| \sim \exp \left( -\frac{1}{2} \sum_{\lambda_k > 0} \lambda_k t \right) \quad (34)$$



along unstable orbits. On the other hand,

$$|\mathcal{A}_n| \sim |t|^{-\nu/2} \quad (35)$$

along stable orbits

The level density of bounded quantum states is described by the trace of the propagator. In semiclassical limit, the trace over the propagator is peaked on around the periodic orbits and stationary saddle points. This allows to semiclassically quantize semiclassical unstable periodic orbits that are densely sited in the invariant set. As a consequence, the semiclassical quantum time delay is

$$\mathcal{T} = \int \frac{d\Gamma_{ph}}{(2\pi\hbar)^{\nu-1}} [\delta(E - H_0 + V) - \delta(E - H_0)] + O(\hbar^{2-\nu}) + 2 \sum_p \sum_a \tau_p^\infty \frac{\cos\left(a \frac{S_p}{\hbar} - \frac{\pi a}{2} \mathbf{m}_p\right)}{\sqrt{|\det(\mathcal{M}_p^a)|}} + O(\hbar) \quad (36)$$

where  $d\Gamma_{ph} = d\mathbf{p}d\mathbf{r}$  and the sum is on all the periodic orbits (primary periodic orbits  $p$  and the number of their repetitions  $a$ );  $S_p(E) = \int \mathbf{p} \cdot d\mathbf{r}$ ,  $\tau_p = \int_E S_p(E)$ ,  $\mathbf{m}_p$  is an index called Maslov index, and  $\mathcal{M}$  is a  $(2\nu - 2) \times (2\nu - 2)$  matrix associated to the Poincaré map in the neighborhood of the  $a$ -orbit. A geodetic equation of a particle on a geometry can be mapped to a problem with an Hamiltonian interaction  $V$ , as done in (36).

Now, let us consider a simplified problem with only  $\nu = 2$  d.o.f, in order to more easily get analytical important proprieties of semiclassical chaotic scatterings and their features. Let us consider a generic projection of our box of geometries to a 2d plane. Now, we study the dynamics in this plane, ignoring the existence of a third dimension. However, we can be so general in our consideration to be practically valid for every chosen projection! Clearly, we remark that we know well how this problem can be only a different simplified problem with respect the 3d one. In this case, the matrix  $\mathcal{M}$  has two eigenvalues:  $\{g_p, g_p^{-1}\}$ , where  $g_p$  is the classical factor  $|g_p| = \exp(\lambda_p \tau_p)$ . As a consequence the complicate equation (36) for the time delay is just reduced to

$$\mathcal{T}(E) = \mathcal{T}_0(E) - 2\hbar \text{Im} \frac{d \ln Z(E)}{dE} + O(\hbar) \quad (37)$$

where  $\mathcal{T}_0(E)$  is the analytical part given by the first integral in (36), while  $Z(E)$  is the Zeta function

$$Z(E) = \prod_p \prod_{a=0}^{\infty} \left( 1 - e^{ia\phi_p} \frac{1}{g_p^a \sqrt{|g_p|}} \right) \quad (38)$$

where

$$\phi_p = \frac{1}{\hbar} S_p - \frac{\pi}{2} \mathbf{m}_p$$

From (37) and (38) one could get, as an application of the Mittag-Leffler theorem, that the pole of the resolvent operators exactly corresponds to the zeros of the Zeta function. In complex energies' plane, the contribution of periodic orbits to the trace of the resolvent operator is related to the Z function by the simple relation

$$\text{tr} \frac{1}{z - H} |_p = \frac{d}{dz} \ln Z(z) = \frac{1}{i\hbar} \sum_p \sum_a \tau_p e^{ia\phi_p} \frac{1}{|g_p|^{a/2}} \quad (39)$$

(we omit extra higher inverse powers of  $|g_p|$ ). But the poles of the resolvent operator and the zeros of the Zeta function are nothing but scattering resonances:

$$Z(E_a) = \mathcal{E}_a - i\Gamma_a/2 = 0$$

Let us comment that if the invariant set contains a single orbit, resonances  $E_a$  satisfy the Bohr-Sommerfeld quantization condition

$$S_p(\mathcal{E}_a) = 2\pi\hbar \left( a + \frac{1}{4} \mathbf{m}_p \right) + O(\hbar^2)$$

while widths satisfy

$$\Gamma_a = \frac{\hbar}{\tau_p} \ln |g_p(\mathcal{E}_r)| + O(\hbar)$$

This last relation is intuitively understood: for a large instability of the periodic orbit  $g_p \gg 1$ , the resonances' lifetime  $\tau_a = \hbar/\Gamma_a \ll 1$ .

Let us return on our general problem, from 2d to 3d. Let us comment that resonances will not always dominate the time evolution of a wavepacket. In fact, in a system like our one, one could expect so many resonances that after the first decays the system will proceed to an average distribution over these resonances' peaks. Considering a wavepacket  $\psi_t(\mathbf{r})$  over many resonances in a region  $W$  in the  $\nu$ -dimensional space, the quantum survival probability is

$$P(t) = \int_W |\psi_t(\mathbf{r})|^2 d\mathbf{r} \quad (40)$$

that can be also rewritten in terms of the initial density operator  $\rho_0 = |\psi_0\rangle\langle\psi_0|$  as

$$P(t) = \text{tr} \mathcal{I}_D(\mathbf{r}) e^{-\frac{iHt}{\hbar}} \rho_0 e^{+\frac{iHt}{\hbar}} \quad (41)$$

where  $\mathcal{I}_D$  is a distribution equal to 1 for  $\mathbf{r}$  into  $D$  while is zero out of the region  $D$ . As done for the time-delay, one can express the survival probability in a semiclassical form

$$P(t) \simeq \int \frac{d\Gamma_{ph}}{(2\pi\hbar)^f} \mathcal{I}_D e^{\mathbf{L}_{cl}t} \tilde{\rho}_0 + O(\hbar^{-\nu+1}) + \frac{1}{\pi\hbar} \int dE \sum_p \sum_a \frac{\cos\left(a\frac{S_p}{\hbar} - a\frac{\pi}{2}\mathbf{m}_p\right)}{\sqrt{|\det(\mathbf{m}_p^a - \mathbf{1})|}} \int_p \mathcal{I}_D e^{\mathbf{L}_{cl}t} \tilde{\rho}_0 dt + O(\hbar^0) \quad (42)$$

where  $\mathbf{L}_{cl}$  is the classical Liouvillian operator, defined in terms of classical Poisson brackets as  $\mathbf{L}_{cl} = \{H_{cl}, \dots\}_{Poisson}$ ;  $\tilde{\rho}_0$  is the Wigner transform of the initial density state

The Sturm-Liouville problem associated to  $\mathbf{L}_{cl}$  defines the Pollicott-Ruelle resonances

$$\mathbf{L}_{cl}\phi_n = \{H_{cl}, \phi_n\}_{Poisson} = \lambda_n \phi_n \quad (43)$$

The eigenstates  $\phi_n$  are Gelfand-Schwartz distributions. They are the ones with unstable manifolds in the invariant set. On the other hand, the adjoint problem

$$\mathbf{L}_{cl}^\dagger \tilde{\phi}_n = \tilde{\lambda}_n \tilde{\phi}_n \quad (44)$$

has eigenstates associated to stable manifolds. The eigenvalues  $\lambda_n$  are in general complex. They have a real part  $\text{Re}(\lambda_n) \leq 0$  because of they are associated to an ensemble bounded periodic orbits. On the other hand  $\text{Im}(\lambda_n)$  describe the decays of the statistical ensembles. One can expand the survival probability over the Pollicott-Ruelle resonances as

$$P(t) \simeq \int \sum_n \langle \mathcal{I}_D | \phi_n(E) \rangle \langle \tilde{\phi}_n(E) | e^{\lambda_n(E)t} | \phi_n(E) \rangle \langle \tilde{\phi}_n(E) | \tilde{\rho}_0 \rangle \quad (45)$$

From this expansion, one can consider the 0-th leading order: it will be just proportional to an exponential  $e^{\lambda_0(E)t}$ . The long-time decay of the system is expected to be related to the classical escape rate  $\gamma(E)$ . So that we conclude that the survival probability goes as  $P(t) \sim e^{-\gamma(E)t}$ , i.e  $s_0 = -\gamma(E)$ .

As a consequence, the cross sections from A to B  $\sigma_{AB} = |\sigma_{AB}|^2$  are dramatically controlled by the Pollicott-Ruelle resonances. let us consider cross sections' autocorrelations

$$C_E(\bar{E}) = \langle \sigma_{BA}(E - \frac{\bar{E}}{2}) \sigma_{AB}(E + \frac{\bar{E}}{2}) \rangle - |\langle \sigma_{BA}(E) \rangle|^2 \quad (46)$$

with  $E$  labelling the energy shell considered. Let us perform the Fourier transform

$$\tilde{C}_E(t) = \int_{-\infty}^{+\infty} C_E(\bar{E}) e^{-\frac{i}{\hbar}\bar{E}t} d\bar{E} \quad (47)$$

As done for the survival probability, we expand (47) all over the Pollicott-Ruelle spectrum so that we obtain

$$\tilde{C}_E(t) \simeq \sum_n \tilde{C}_n \exp(-\text{Re}\lambda_n(E)t) \cos \text{Im}\lambda_n(E)t \quad (48)$$

where  $\tilde{C}_n$  are coefficients of this expansion. In particular the leading order of (48) is related to (45) for  $\text{Im}\lambda_0 = 0$ :

$$\tilde{C}_E(t) \simeq \exp(-\gamma(E)t) \quad (49)$$

corresponding to the main Lorentzian peak

$$C_E(\bar{E}) \sim \frac{1}{\bar{E}^2 + (\hbar\gamma(E))^2} \quad (50)$$

while (48) corresponds to a spectral correlation

$$C_E(\bar{E}) \simeq \sum_n \left\{ \frac{C_n}{(\bar{E} - \hbar Im\lambda_n)^2 + (\hbar Re\lambda_n)^2} + \frac{C_n}{(-\bar{E} - \hbar Im\lambda_n)^2 + (\hbar Re\lambda_n)^2} \right\} \quad (51)$$

We conclude resuming that a semiclassical quantum chaotic scattering approach leads to following conclusions about the box of geometries problem: i) the existence of chaotic regions of trapped trajectories has to be a consequence of our scattering problem; ii) the qualitative behavior of survival probability and correlation function is qualitatively understood as a decreasing function in time with an exponent determined by classical chaos scattering considerations.

### 3.2.1. Quantum field theories

In this section, we will formally discuss the problem of scattering from a QFT point of view. This is based on the path integral approach on  $N$  geometries. In the path integral integration, one will start to "explore" field configurations with energies comparable to the inverse asperities' size. In the fourier transform space, field configurations will be prevalently scattered by asperities if the a system has energy comparable to the inverse asperities' size. We are against a chaotic quantum field theory problem. In a chaotic quantum field theory, there are not trapped trajectories in space-time but there trapped configurations in the infinite dimensional space of fields! In analogy to semiclassical chaotic non-relativistic quantum mechanics, one can consider a semiclassical approximation in a regime in which the fields' action is much higher than  $\hbar$ :  $I \gg \hbar$ . In this approximation, we have a formal understanding of the chaotic quantum field theory problem. The corresponding WKB propagator for a quantum field has a formal expression

$$\langle \phi_0, t_0 | \phi_1, t_1 \rangle \simeq \sum_n \mathcal{A}_n(\phi_0, t_0 | \phi_1, t_1) e^{\frac{i}{\hbar} I_n} \quad (52)$$

where we are summing on all over the classical orbits in the fields' configurations' space, while amplitudes  $\mathcal{A}_n$  are

$$\mathcal{A}_n(\phi_0, t_0 | \phi_1, t_1) = \frac{1}{(2\pi i \hbar)^{\nu/2}} \sqrt{|det[\partial_{\phi_0} \partial_{\phi_0} I_n[\mathbf{r}, \mathbf{r}_0, t]]|} e^{-\frac{i\pi h_n}{2}} \quad (53)$$

where  $h_n$  counts the number of conjugate points along the  $n$ -th orbits.

We will expect that all rigorous results obtained in literature of classical chaotic scatterings, about the existence of invariant set with their topological robust proprieties discussed in part above, will be not rigorously extended for an infinite dimensional space of fields. A complete theory regarding these aspects in QFT is not known to me. Nevertheless let us intuitively think something similar happen in space of fields, even if more complicated. The presence of chaotic zones of trapped periodic fields' configurations in a subregion of the configurations' space, corresponding to the one confined into our system, is expected for our problem. Also for fields, chaotic unstable trajectories in the fields' space are expected, as well as a large number of fields' resonances in QFT S-matrices, generalizing Pollicot-Ruelle ones. The survival probability for a field are expected to exponentially decrease as in semiclassical quantum mechanical case.

On the other hand, a general space of different fields, the presence of interaction terms in the lagrangian leads to tree-level transitions' processes that has to be considered as leading orders in the semiclassical saddle point perturbative expansion. As a consequence, chaotic fields' trapped trajectories have to be thought as a multifields' ones. The result can be imagined as a chaotic cascade of processes among fields, in which a part of different fields are trapped in the system, interacting and scatterings and decaying each others. For example, let us imagine one pure electromagnetic wave entering inside the box of geometries. This starts to be diffracted into different direction, so that initial coherent photons will start to re-meet each other in a different state. Of course, if their energy is enough, they can produce couples of  $e^+e^-$ ,  $q\bar{q}$  and so on. Then, these fields will interact each other through electromagnetic, strong and weak interactions. The final system will be full of new fields, and it will have highly chaotic trapped zone.

An alternative formal way it the following. Suppose interdistances much higher than geometries' dimensions. This case is a simplified one with respect to the realistic problem. In this case, we can define a transition amplitude for each geometry. Let us suppose to be interested to calculate the transition amplitude for a field configuration  $\phi_0$  to a field configuration  $\phi_N$ .  $\phi_0$  is the initial field configuration defined on a  $t_0$ , before entering in the system, while  $\phi_N$  is a field configuration of a time  $t_N$ , corresponding to a an out-going state from the system. For simplicity, we can

formalize the simplified problem as a 4D-box, with  $n \times m \times p$  singularities in 3D,  $n$  in the x-axis,  $m$  in y-axis,  $p$  in z-axis (not necessary disposed as a regular lattice). Let us call  $\mathcal{N}_1, \mathcal{N}_2$  the sides sited in the xy-planes,  $\mathcal{M}_{1,2}$  in xz-planes,  $\mathcal{P}_{1,2}$  in zy-planes, delimiting the 3D-space-box. Let us consider an incident field  $\phi_0$  on the 2D plane  $\mathcal{N}_1$ , with  $n \times m$  singularities. Then the  $n \times m$  singularities will scatter the incident field in  $n \times m$ -waves. From each diffractions, the out-waves will scatter on a successive singularity, penetrating in the box, or to the other nodes in the same plane  $\mathcal{N}_1$ , and so on. Our problem is to evaluate the S-matrix from the in-state 0 to the out-the box state. One will expect that a fraction of initial probability density will escape from the 3D box by the sides  $\mathcal{N}_{1,2}, \mathcal{M}_{1,2}, \mathcal{P}_{1,2}$ , another fraction will be trapped "forever" (for a time-life equal to the one of the system) inside the box. As a consequence, one has to consider all possible diffraction stories or diffraction paths. Clearly, one has also to consider paths in which the initial wave goes back and forth in the system before going-out.

One example of propagation Path  $0 - 111 - 222 - 333 - \dots - nmp - N$

$$\langle \phi_0, t_0 | \phi_{111,in}, t_{111,in} \rangle \langle \phi_{111,in}, t_{111,in} | \phi_{111,out}, t_{111,out} \rangle \langle \phi_{111,out}, t_{111,out} | \phi_{222,in}, t_{222,in} \rangle \quad (54)$$

$$\times \langle \phi_{222,in}, t_{222,in} | \phi_{222,out}, t_{222,out} \rangle \dots \langle \phi_{(n-1,m-1,p-1)}, t_{(n-1),(m-1),(p-1)} | \phi_{nmp}, t_{nmp} \rangle \langle \phi_{n,m,p}, t_{n,m,p} | \phi_N, t_N \rangle$$

where  $|\phi_{ijk,in}, t_{ijk,in}\rangle$  and  $|\phi_{ijk,out}, t_{ijk,out}\rangle$  are states before and after entering in the horizonless geometry  $ijk$ . In order to evaluate  $\langle \phi_0, t_0 | \phi_{nmp}, t_{nmp} \rangle$  one has to consider all the possible propagation paths from the initial position to the  $nmp$ -th singularity<sup>12</sup>. We define these amplitudes as

$$\langle \phi_{ijk}, t_{ijk} | \phi_{i'j'k'}, t_{i'j'k'}, in \rangle = \int_{\mathcal{M}_0} \mathcal{D}\phi e^{iI[\phi]} \quad (55)$$

while

$$\langle \phi_{ijk,in}, t_{ijk,in} | \phi_{ijk,out}, t_{ijk,out} \rangle = \int_{\mathcal{M}_{ijk}} \mathcal{D}\phi e^{iI[\phi]} \quad (56)$$

where  $\mathcal{M}_0$  is the Minkowski space-time, while  $\mathcal{M}_{ijk}$  is the  $ijk$ -cone space-time. Again one can easily get that for a large system of naked singularities, it will exist a class of propagators' paths, reaching the out state  $|\phi_N, t_N\rangle$  only for a time  $t_N \rightarrow \infty$ . A simple example can be the propagator paths

$$|\langle \phi_{ijk}, t_{ijk} | \phi_{i'j'k'}, t_{i'j'k'}, in \rangle|^2 |\langle \phi_{ijk}, t_{ijk}^{(1)} | \phi_{i'j'k'}, t_{i'j'k'}^{(1)} \rangle|^2 \dots |\langle \phi_{ijk}, t_{ijk}^{(\infty)} | \phi_{i'j'k'}, t_{i'j'k'}^{(\infty)} \rangle|^2 \quad (57)$$

where  $t_{ijk}^{(\infty)} > \dots > t_{ijk}^{(1)} > t_{ijk}$  and  $t_{i'j'k'}^{(\infty)} > \dots > t_{i'j'k'}^{(1)} > t_{i'j'k'}$ . This amplitude is non-vanishing in such a system as an infinite sample of other ones. We can formally group these propagators in a  $\langle BOX | BOX \rangle$  propagator, evaluating the probability that a field will remain in the box of singularities after a time larger than the system life-time. On the other hand, let call  $\langle BOX | OUT \rangle$  and  $\langle OUT | OUT \rangle$  the other processes.

Considering interactions, one will also use S-matrices. We can write a generic S-matrix for one diffraction path as

$$\langle in | S^{Kth} | out \rangle = S_{0-1jk} S_{ijk} S_{i'j'k'} \dots S_{(i^{n-1}j^{m-1}k^{p-1})-(i^n j^m k^p)} \quad (58)$$

A class of paths like with conditions

$$i \leq i' \leq i+1 \quad (59)$$

$$j \leq j' \leq j+1 \quad (60)$$

$$k \leq k' \leq k+1 \quad (61)$$

...

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<sup>12</sup> Propagators inside singular curved geometries like horizonless conic singularities mathematically exist and they were discussed in papers [64]. Other discussions about scalar fields on Kasner space-time can be found in [68]. Contrary to the definition of propagators in quantum space-time foams, in our case the N geometries are well defined and continuos. As a consequence one can define the propagation of fields inside them. Problematic regions are discontinuous ones among the geometries. In principle, one has to define also propagators for these linking regions, with an opportune smoothing procedure applied on these regions.

$$i^{n-1} \leq i^n \leq i^{n-1} + 1 \quad (62)$$

$$j^{m-1} \leq j^m \leq j^{m-1} + 1 \quad (63)$$

$$k^{p-1} \leq k^p \leq k^{p-1} + 1 \quad (64)$$

We call these class of paths "minimal paths". In fact, in these paths there are not back-transitions. The total number of "minimal paths" is  $n \times m \times p \times (n-1)$ . On the other hand, the number of paths with back and forth scatterings will diverge.

As a consequence, the total S-matrix is the sum over all possible infinite diffraction paths

$$\langle in | S_n^{OUT} | out \rangle = \sum_{paths} \langle in | S_n^{K-th} | out \rangle \quad (65)$$

accounting for all the paths leading from the in-state to the out-of-box state.

For a completeness of our discussion, let us reformulate the non-relativistic quantum problem in a non-relativistic path integral formulation. We will use here the bracket-notation, in which the propagator from  $(x_0, t_0)$  to  $(x_1, t_1)$  is

$$K(x_0, t_0; x, t_1) = \langle x_0, t_0 | x_1, t_1 \rangle$$

This will be equivalent to wave functions' formulation considered in section 3.2. In this case, a problem of  $\langle OUT | OUT \rangle$  is reformulated not with propagators in the fields' space but in the same space-time points.  $\langle OUT | OUT \rangle$  will account for all possible paths leading to an in-coming state  $|x_0, t_0\rangle$  to another state out of the box. Again, such a problem is chaoticized by the fact that one has to consider the interference of all possible paths passing for all possible horizonless geometries. A simple example of a path inside the OUT-OUT class of paths is  $0 - 111 - 222 - 333 - \dots - nmp - N$

$$\begin{aligned} & \langle x_0, t_0 | x_{111,in}, t_{111,in} \rangle \langle x_{111,in}, t_{111,in} | x_{111,out}, t_{111,out} \rangle \langle x_{111,out}, t_{111,out} | x_{222,in}, t_{222,in} \rangle \\ & \times \langle x_{222,in}, t_{222,in} | x_{222,out}, t_{222,out} \rangle \dots \langle x_{(n-1,m-1,p-1)}, t_{(n-1),(m-1),(p-1)} | x_{nmp}, t_{nmp} \rangle \end{aligned} \quad (66)$$

where  $|x_{ijk,in}, t_{ijk,in}\rangle$  and  $|x_{ijk,out}, t_{ijk,out}\rangle$  are states before and after entering in the geometry  $ijk$ .

One can find trapped propagators like

$$|\langle x_{ijk}, t_{ijk} | x_{i'j'k'}, t_{i'j'k'} \rangle|^2 |\langle x_{ijk}, t_{ijk}^{(1)} | x_{i'j'k'}, t_{i'j'k'}^{(1)} \rangle|^2 \dots |\langle x_{ijk}, t_{ijk}^{(\infty)} | x_{i'j'k'}, t_{i'j'k'}^{(\infty)} \rangle|^2 \quad (67)$$

where  $t_{ijk}^\infty > \dots > t_{ijk}^{(1)} > t_{ijk}$  and  $t_{i'j'k'}^\infty > \dots > t_{i'j'k'}^{(1)} > t_{i'j'k'}$ . A class of paths from OUT to BOX state will be attracted in these trapped paths.

#### 4. CONCLUSIONS AND OUTLOOKS

In this paper, we have shown how a semiclassical black hole can be obtained as a system of naked singularities (even if probably smoothed at the Planck length). In particular, we have mathematically defined a new object called frizzyball. A frizzyball emits a quasi-thermal radiation. Deviations from thermality are related to the particular disposition of geometries, so that they carry small informations about the space-time structure. In such a system, we have shown how infalling wave functions will be inevitably chaoticized. These chaotic effects will manifest themselves before ingoing into the full transplanckian regime. These statements are sustained from the analysis of the non-relativistic scattering problem of a particle ingoing to a system of  $N$  horizonless geometries.

This system is a Sinai Biliard of the space-time topology. We have argued how trapped chaoticized zones will be formed among the space-time asperities. However, the frizzy topology is sustained by the frizzy-ball mass. But the mass is gradually lost in quasi Bekenstein-Hawking evaporation. As argued, at a certain critical mass, a phase transition of the space-time topology into a trivial Minkowski vacuum state is expected. During this process, asperities are gradually washed-out and trapped information are gradually leaked in the external environment. The final evaporation will release all hidden information. We have called this phenomena "final information burst". As a consequence, the S-matrix  $\langle collapse | S | total\ evaporation \rangle$  is unitary and well define, without any paradoxes. From the point of view of an external observer, for  $M > M_\odot$  such a process will be longer than the age of the Universe

( $t_{BH} > 10^{74}$  s), while from the reference frame of an internal trapped particle, the time will be very short  $t_{BH} \ll 1$  yr (depending by the frizzy-ball mass). A part of the information will be trapped by semiclassical black hole just because of the causal structure of the interior, so our proposal seems un-useful from this point of view. However, there is an important difference, that can be understood as follows. Let us consider an entangled Hawking's pair, one outgoing and the other one ingoing. In a semiclassical black hole, they will remain entangled and the smoothed causal structure cannot disentangle them. On the other hand, if as suggested the infalling one will start to be scattered in the Sinai biliard, then it will start to be very efficiently converted in a shower of particles, as shown above. As a consequence, the initial entanglement is chaotically lost because of quantum decoherence: the out-going pair is entangled with a large number of fields inside the system. The efficiency of this process will exponentially increase with the number of infalling particles. This is a new quantum decoherence effect induced by the space-time topology. Let us note that the causal structure of a frizzlyball has a non-trivial topology. As a consequence, the associated Penrose' diagram is a complicated superposition of Penrose' diagrams of the different metrics.

At this point, I am tempted to suggest that black holes and frizzlyballs could be observationally differentiated through their gravitational lensing proprieties. In [65, 66], the difference between gravitational lensing signatures of black holes and naked singularities were discussed in details. In a broad sense, frizzlyballs are new solutions interpolating among black holes' and naked singularities' ones. In fact, the asperity parameter  $\sigma_\beta^2$  defined in section 2 is connected to geometric standard deviations of the frizzlyballs' surfaces  $\sigma_r^2$  ( $r$  bh radius) with respect to a (semi)classical black hole. In our model,  $\sigma_r$  is a free parameter in a large range  $l_{Pl} < \sigma_r < r_S$  ( $l_{Pl}$  is the Planck lenght,  $r_S$  the Schwarzschild radius).  $\sigma_r \leq l_{Pl}$  corresponds to a black hole while  $\sigma_r \geq r_S$  to separated naked singularities. However, in the framework of our semiclassical effective model, we cannot establish the largeness of such a parameter. If such a parameter is determined by an UV completion of our model and/or if it depends on initial conditions of star collapses is still unclear. In fact, these problems could be connected to other deep issues. For instance, the validity of (weak or strong) cosmic censorship conjectures and hoop conjectures still remains unclear. In fact, see references [69] for an overview of discussions on the cosmic censorship conjectures and their possible violations; see references [70] about the hoop conjecture. If frizzlyballs exist, these conjectures will have to be considered true only as *approximated/accidental* ones. Anyway, the dynamical reason of such conjectures remains unknown. A possibility could be that frizzlyball could be formed through a *dynamical censorship mechanism*. We define *dynamical censorship mechanism* as a generic evolution from a configuration of separated naked singularities to a frizzlyball. Separated naked singularities could be destabilized by external electromagnetic and/or gravitational and/or matter fields' perturbations. These kinds of instabilities at classical level, for Super-extremal Kerr naked singularities  $J^2 > M > 0$  ( $J$  bh spin parameter,  $M$  bh mass) and Super-extremal Reissner-Nördstrom bh  $|Q| > M > 0$  ( $Q$  bh charge), were discussed in [72]. Because of that, formation of frizzlyballs could be energetically convenient, as (meta)stable configurations against perturbations. Evidences from numerical simulations, that naked singularities can be formed in collapses [71], seem to sustain our hypothesis. If an unstable naked singularity was formed, it would decay to a system of naked singularities disposed as a frizzlyball. As a consequence, a frizzlyball with  $\sigma_r \gg l_{Pl}$  could be detectable through its gravitational lensing characteristics. These aspects deserve future studies beyond the purpose of this paper.

To conclude, the black hole interior could be frizzy rather than smoothed as thought in semiclassical approach, even at smaller energy scales compared to the Planck scale. This will induce a high chaotization of infalling information. We are not proposing a final solution about BH problems, but a possible different point of view on these issues.

I think that this hypothesis deserves future investigations by different communities of physicists, from chaos theory and (classical and quantum) gravity theory <sup>13</sup>.

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<sup>13</sup> For other discussions on these aspects see [73] .



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